A FEEDBACK EXTENSION TO THE NUMERICAL SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS

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1. Abstract

A feed-back extension procedure is developed for the numerical solution of a class of nonlinear boundary value problems associated with anti-plane shear or Hencky's theory of plasticity. This extends previous results using dimensional reduction in energy-asymptotic format.[1]

2. Introduction

In an earlier paper [4], the method of dimensional reduction for quasilinear boundary value problems was introduced. A generalisation was proposed which allows for the possibility of different order of dimensionally reduced models in different parts of the underlying domain. This paper is an attempt to fulfill that promise with the purpose of making the method of dimensional reduction still more efficient and robust.

As in [4], the basic idea is to find a minimiser u_N of the given energy functional in a proper subspace V_N which is characterised by the basis functions $\{\psi_j\}_{j=0}^{N}$:

$$V_N = \{\Sigma_{j=0}^{N} c_j(\xi)\psi_j(\eta)\}$$

where $\xi = x_1 \in [0,1], \eta = x_2/d, x_2 \in [-d,d]$, and d denotes the half thickness of the domain. Thus the model of order N of reduced dimension was introduced. See [4] for the choice of $\{\psi_j\}_{j=0}^{N}$ and related convergence properties (optimal rates) as $d \downarrow 0$ or $N \to \infty$.

Due to the singularities which can stem from the loading or the presence of corners, it is necessary (for efficiency and accuracy) to be able to introduce higher order models near these layers only. In this paper we propose a feed back extension procedure that facilitates this by allowing different orders N_i in different parts of [0,1].

3. Notation and Model Problem

We shall confine our study to the following class of problems. Find $u \in W$ such that

$$\forall v \in W, \qquad Au(v) = G(v) \tag{3.1}$$

where

$$Au(v) = \int_\Omega F(|\nabla_d u|^2)\nabla_d u \cdot \nabla_d v \, d\xi \, d\eta \tag{3.2}$$

$$G(v) = d^{1-\mu}\int_0^1 \beta(\xi)[v(\xi,1) + v(\xi,-1)] \, d\xi \tag{3.3}$$

$$F(t) = 1 + t^n, \; n \in \mathbf{N}, \; t \in \mathbf{R}\setminus\mathbf{R}_- \tag{3.4}$$

$$\mu \begin{cases} \in \mathbf{R} \text{ characterizes three asymptotic ranges of loads : } \beta d^{-\mu} \\ \text{– such that the limit traction on } \Gamma_\pm \text{ as } d \downarrow 0 \\ \text{is 0 for } \mu < 0, \text{ finite for } \mu = 0, \text{ and infinite for } \mu > 0 \end{cases} \tag{3.5}$$

$$\Omega =]0,1[\times]-1,1[\tag{3.6}$$

$$\Gamma_0 = (\{0\}\times]-1,1[) \cup (\{1\}\times]-1,1[) \tag{3.7}$$

$$\Gamma_+ = [0,1] \times \{1\} \tag{3.8}$$

$$\Gamma_- = [0,1] \times \{-1\} \tag{3.9}$$

$$W = W_{(0)}^{1,2n+2}(\Omega) = \{v \in W^{1,2n+2}(\Omega); v|_{\Gamma_0} = 0\} \tag{3.10}$$

$$\nabla_d = \left(\frac{\partial}{\partial \xi}, \frac{1}{d}\frac{\partial}{\partial \eta}\right) \tag{3.11}$$

This scalar problem corresponds to finding a minimiser in W for the energy in anti-plane shear in finite elasticity, [6] and [7] and the torsion problem for a bar, see [8] and [9]. See [4].

We define the dimensionally reduced solution of order N to be the solution u_N in $V_N \subset W$ for which

$$\forall v \in V_N \subset W, \qquad Au_N(v) = G(v) \tag{3.12}$$

given V_N a subspace in W of the form

$$V_N = \{v \in W : v(\xi,\eta) = \Sigma_{j=0}^{N} c_j(\xi)\psi_j(\eta)\} \tag{3.13}$$

The family of subspaces $\{V_N\}_{N=0}^{\infty}$ *is* characterised by the choice of $\{\psi_j\}_{j=0}^{\infty}$ called the basis or Ansatz functions.

In [4] these basis functions were selected to yield the optimal rate of convergence of $\|u - u_N\|_{H^1}$ as $d \downarrow 0$. We thus had to select ψ_j to be a polynomial of degree 2j. Importantly, the same choice is valid for all three ranges of loads (three signs of μ in (3.5) and (3.3)). For F in (3.4) depending on η, [4] indicated that the same procedure would yield ψ_j to be a nonpolynomial solution of a second order Sturm Liouville problem. (See also [5, Remark 3.9].)

Let $\overline{u_N}$ be the Nth partial sum in the formal asymptotic expansion as given in [4]. Let D_1 be the operator defined by $D_1 u = \frac{d^2 u}{d\xi^2}$ mapping $Dom(D_1) = W^{2,2n+2}(0,1) \cap W_0^{1,2n+2} \to L^{2n+2}(0,1)$. We got for $\mu \leq 0$.

Theorem 3.1 *Let* $\mu \leq 0$ *and* $n \in \mathbf{Z}_+$. *Let* $u, \overline{u_N} \in W^{1,\infty}$ *be bounded there independently of d. Let* $\beta \in Dom(D_1^N)$. *Then there exists* C_N *independent of d such that*

$$\|u - u_N\|_{H^1} \leq C_N d^{2N+\frac{3}{2}-\mu}$$

Since, for a given practical problem, we cannot depend on d being sufficiently small to ensure that a given tolerance criterion can be satisfied via the previous theorem, we have considered in [4] to increase N. Again, optimal rates in this scenario (d fixed, N increasing) were established in [4]. From the computational experience in [4] and elsewhere, it became clear that it was unnecessary (read: wasteful) to increase N uniformly everywhere in [0,1[. Rather, there were clearly defined layers (near the boundary and/or rough spots in the load). We propose to increase N near these layers only as our extension procedure.

Let $\mathrm{I} = (0,1) = \cup_{i=1}^{m}\mathrm{I}_i$, and $\mathrm{I}_i \cap \mathrm{I}_j = \emptyset, i \neq j \; \forall i,j \in [1,m]$. Let $\mathcal{N} = (N_i)_{i=1}^{m}$ be an m-vector of nonnegative integers (N_i = no. of basis functions used in I_i). Consider

$$\begin{aligned} V_{\mathcal{N}} = \{v : \quad & v(\xi,\eta) = \Sigma_{j=0}^{N} v_j(\xi)\psi_j(\eta) \text{ such that} \\ & N = \|\mathcal{N}\|_\infty, \quad v_j(\xi) = 0 \text{ for } \xi \in \cup_{j>N_i}\mathrm{I}_i\} \end{aligned} \tag{3.14}$$

a subspace of V_N. Solving

$$\forall v \in V_{\mathcal{N}} \subsetneq V_N, \quad Au_{\mathcal{N}}(v) = G(v) \tag{3.15}$$

for $u \in V_{\mathcal{N}}$ is the generalised dimensionally reduced Galerkin problem.

A key ingredient in the selection of the distribution of orders $\mathcal{N}$ – the local a posteriori estimators – will be developed in the following section.

4. Local A Posteriori Error Estimators

Define the estimator for (0,1) *and order* $\mathcal{N}$ *as*

$$Est(\mathcal{N}) = \left\|\frac{1}{d}\frac{\partial e}{\partial \eta}\right\|_{L^2(\Omega)} \tag{4.1}$$

where $e \in H_{(0)}^{1}(\Omega)$ is the solution of

$$\forall v \in H_{(0)}^{1}(\Omega) : \int_{-1}^{1}\int_{0}^{1} \frac{1}{d}\frac{\partial e}{\partial \eta}\,\frac{1}{d}\frac{\partial v}{\partial \eta} d \, d\xi \, d\eta = G(v) - Au_{\mathcal{N}}(v) \tag{4.2}$$

the right hand side being the residual $(Au - Au_{\mathcal{N}})(v)$. Although e is not well defined, $\frac{\partial e}{\partial \eta}$ and $Est(\mathcal{N})$ are, provided the following solvability condition is satisfied

$$\begin{aligned} \forall c \in H^1(0,1) : \quad & \int_0^1 \beta(\xi)2c(\xi)d^{1-\mu} \, d\xi = \\ & \int_{-1}^{1}\int_{0}^{1} F(|\nabla_d u_{\mathcal{N}}|^2)\frac{\partial u_{\mathcal{N}}}{\partial \xi}c'(\xi) \, d \, d\xi d\eta \end{aligned} \tag{4.3}$$

However, this is satisfied (even for $c \in W_0^{1,2n+2}$) if

$$1 \in span(\{\psi_j\}_{j=0}^{N}) \tag{4.4}$$

cf. (3.12) and (3.13). This condition is met for any choice of basis functions with optimal rates, see [4].

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Similarly define the local error estimator

$$Est_i(\mathcal{N}) = \left[\int_{-1}^{1}\int_{I_i}\left(\frac{1}{d}\frac{\partial e}{\partial \eta}\right)^2 d\, d\xi d\eta\right]^{\frac{1}{2}}, 1 \le i \le m \tag{4.5}$$

As in [2] we define upper (lower) error estimator to mean

$$\|u - u_{\mathcal{N}}\|_{H^1} \le (\ge) Est$$

Theorem 4.1 *Let u and $u_{\mathcal{N}}$ be the exact and dimensionally reduced solutions (see (3.1) and (3.15)). Then Est as defined in (4.1) is an upper estimator, i.e.*

$$\|u - u_{\mathcal{N}}\|_{H^1} \le Est(\mathcal{N})$$

Proof: Bound from above and below $(Au - Au_{\mathcal{N}})(u - u_{\mathcal{N}})$. ////

In the language of [3], Est is a guaranteed U-estimator (G-estimator). Another attractive property of Est is that $Est/\|u - u_{\mathcal{N}}\|_{H^1}$ tends to 1 as d tends to zero for $\mu < 0$ and β sufficiently smooth.

Theorem 4.2 *Let u and $u_{\mathcal{N}}$ be the exact and dimensionally reduced solutions (cf. (3.1) and (3.15)) and have gradients bounded uniformly in d. Let $\mu < 0$ and $\beta \in Dom(D_1^N)$. Then Est as defined in (4.1) is asymptotically exact:*

$$Est(\mathcal{N}) = \|u - u_{\mathcal{N}}\|_{H^1}(1 + O(d))$$

Proof: Due to restrictions in length, we merely mention that one can establish

$$Est(\mathcal{N}) \le \|u - u_{\mathcal{N}}\|_{H^1}\frac{\|e\|_{H^1}}{\|\frac{1}{d}\frac{\partial e}{\partial \eta}\|_{L^2}}(1 + O(d))$$

and bound the middle factor on this right hand side from above. ////

5. Computational Aspects

From an implementational point of view, the nice mathematical properties of Est established in the previous section will not suffice, since finding e as a solution of a second order O.D.E. might be too costly. Therefore we give now some formulae that can be used to compute e and Est in practice.

First, let us introduce a basis in $L^2(-1,1)$ with which we will work:

$$\begin{aligned} \phi_0(\eta) &= 1 \\ \phi_j(\eta) &= \int_{-1}^{\eta} l_{j-1}(t)\, dt \text{ for } j \ge 1 \end{aligned} \tag{5.1}$$

where l_j is the jth Legendre polynomial.

Lemma 5.1 *The following formulae for e defined in (4.2) hold:*

$$e = \Sigma_{j=0}^{N+2} e_j \phi_j$$

and furthermore

$$e_j = \begin{cases} 0 & \text{for } 1 \le j \le N \\ d^2 \int_{-1}^{1} Lu_{\mathcal{N}} \phi_j\, d\eta / \int_{-1}^{1} (\phi_j')^2\, d\eta & \text{for } j = N+1, N+2 \end{cases}$$

where $Lu = \nabla_d[F(|\nabla_d u|^2)\nabla_d u]$.

Proof: $e \in V_{N+2}$ so the first assertion is clear and $\frac{\partial e}{\partial \eta} = \Sigma_{j=1}^{N+2}\phi_j'(\eta) e_j$ and if we denote by $(\cdot,\cdot)$ the inner product in $L^2(-1,1)$ we have $\forall z \in W_0^{1,2n+2}$:

$$\begin{aligned} (e_j, z)\int_{-1}^{1}(\phi_j')^2\, d\eta &= \int_{-1}^{1}\left(\frac{\partial e}{\partial \eta}, \phi_j' z\right) d\eta \\ &= \int_{-1}^{1}(\phi_j' e_j, \phi_j' z)\, d\eta \\ &= d(Au - Au_{\mathcal{N}})(\phi_j z) \\ &= 0 \text{ for } 1 \le j \le N \end{aligned}$$

Hence, $e_j = 0$ for $1 \le j \le N$ and the two remaining components are obtained in a similar fashion using now the definiton of A in the next to last line and the fact $\phi_j(1) = \phi_j(-1) = 0$ for $j > 1$. ////

The following formulae derived from the previous ones are still more useful:

Lemma 5.2 *There exists a constant matrix A such that*

$$\begin{pmatrix} e_{N+1} \\ e_{N+2} \end{pmatrix} = A \begin{pmatrix} 2\beta d^{1-\mu} - F(|\nabla_d u_{\mathcal{N}}|^2)\frac{1}{d}\frac{\partial u_{\mathcal{N}}}{\partial \eta}|_{-1}^{1} \\ -F(|\nabla_d u_{\mathcal{N}}|^2)\frac{1}{d}\frac{\partial u_{\mathcal{N}}}{\partial \eta}\phi_1|_{-1}^{1} \end{pmatrix}$$

where e_{N+1}, e_{N+2} were defined in the previous Lemma.

Proof: Consider

$$-Lu_{\mathcal{N}} = \frac{\partial^2 e}{\partial \eta^2}\frac{1}{d^2} = \frac{1}{d^2}\frac{\partial^2}{\partial \eta^2}(e_{N+1}\phi_{N+1} + e_{N+2}\phi_{N+2})$$

We denote by $(\cdot,\cdot)$ the inner product in $L^2(0,1)$. Now let $v = \phi_0 z$, then

$$\begin{aligned} 0 &= (\beta d^{1-\mu}, (\phi_0(1) + \phi_0(-1)z) - Au_{\mathcal{N}}(\phi_0 z) \\ &= (2\beta d^{1-\mu}, z) + ((Lu_{\mathcal{N}}, \phi_0), z) - \left(F(|\nabla_d u_{\mathcal{N}}|^2)\frac{1}{d}\frac{\partial u_{\mathcal{N}}}{\partial \eta}|_{-1}^{1}, z\right) \\ &= \left(2\beta d^{1-\mu} - B_{11}e_{N+1} - B_{12}e_{N+2} - F(|\nabla_d u_{\mathcal{N}}|^2)\frac{\partial u_{\mathcal{N}}}{\partial \eta}|_{-1}^{1}, z\right) \end{aligned}$$

Next letting $v = \phi_1 z$, we obtain similarly

$$0 = -\left(B_{21}e_{N+1} + B_{22}e_{N+2} + F(|\nabla_d u_{\mathcal{N}}|^2)\frac{\partial u_{\mathcal{N}}}{\partial \eta}\phi_1|_{-1}^{1}, z\right)$$

where

$$B_{ij} = \int_{-1}^{1}\frac{1}{d^2}\frac{\partial^2}{\partial \eta^2}\phi_{N+j}\phi_{i-1}\, d\, d\eta$$

for $1 \le i, j \le 2$. The matrix B is invertible into A. ////

We next introduce the heuristic principle which will guide us to an efficient extension procedure based on the local a posteriori error estimators.

Heuristic 5.1 *Let the error associated with the generalized dimensional reduction be estimated by*

$$(\Sigma_{i=1}^{m} Est_i^2(N_i))^{\frac{1}{2}}$$

and the cost (work) be estimated by

$$\Sigma_{i=1}^{m} W(N_i, I_i)$$

Then we aim at achieving

$$Est_i^2(N_i) - Est_i^2(N_i - 1) \propto W(N_i, I_i) - W(N_i - 1, I_i)$$

by increasing N_i by 1 where the error-cost quotient is maximal.

Reasoning: Minimizing the error at fixed cost with respect to N_i, yields via Lagrange's multiplier and a backwards difference approximation the proportionality aimed at in the Heuristic. ////

A typical choice for workestimate is

$$W(N_i, I_i) = (\alpha_1 N_i + 1)^{\alpha_2}|I_i| \tag{5.2}$$

for some choice of positive α_i, i=1,2.

Note that we selected the prime functions of Legendre polynomials as basis merely to be able to establish computational formulae; a change of basis within the same span merely requires a linear transformation in order to modify the formulae for the new choice of basis functions.

From the computational point of view it is rather important exactly which basis functions one selects (this has to be done hierarchically).

6. Choice of Basis Functions

Let $\Psi_N = (\psi_j)_{j=0}^{N}$ and $U = (u_j)_{j=0}^{N}$ such that

$$u_N = U \cdot \Psi_N$$

The generalised Galerkin problem (3.15) transforms to the following system of O. D. E.s:

$$-\frac{d}{dx}d(P(U,U')U') + \frac{1}{d}Q(U,U')U = R \tag{6.1}$$

where P and Q are matrices defined by:

$$P_{ij} = \int_{-1}^{1} F(|\nabla_d u_N|^2)\psi_i\psi_j\, d\eta \qquad (6.2)$$

$$Q_{ij} = \int_{-1}^{1} F(|\nabla_d u_N|^2)\frac{1}{d^2}\psi_i'\psi_j'\, d\eta \qquad (6.3)$$

for $1 \le i, j \le N$. Since the system (6.1) is hard to analyse in its nonlinear form, we will bracket with linear ones. If $\|\nabla_d u_N\|_\infty \le M$, then

$$(P^{lin}U', U') \le (P(U, U')U', U') \le (1 + M^{2n})(P^{lin}U', U')$$

$$(Q^{lin}U', U') \le (Q(U, U')U', U') \le (1 + M^{2n})(Q^{lin}U', U')$$

where P^{lin} and Q^{lin} are defined as in (6.2) except with $F \equiv 1$. This allows us to analyse some of the behavior of (6.1).

An elementary Saint-Venant like principle holds for a related linear boundary value problem posed over the semi-infinite strip: $\Omega^\infty =]0, \infty[\times] -1, 1[$ with boundaries $\Gamma_0^\infty = \{0\}\times] -1, 1[$, and $\Gamma_\pm^\infty$ which is defined analogously to (3.8) and (3.9) respectively. The function

$$u(x, y) = \Sigma_{k=0}^{\infty} a_k \cos\frac{k\pi y}{d} \exp -\frac{k\pi x}{d}$$

is the solution of

$$\begin{aligned} \Delta_d u &= 0 \quad \text{in } \Omega^\infty; \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_\pm^\infty; \\ u &= g \quad \text{on } \Gamma_0^\infty \end{aligned}$$

Here $\lambda_k = \frac{k^2\pi^2}{d^2}$, $k \in \mathbb{N}_0$ are the eigenvalues corresponding to the eigenfunctions given by the following B.V.P. (in the y-direction)

$$\phi_k'' + \lambda_k\phi_k = 0 \text{ in }]-d, d[$$

$$\phi_k' = 0 \text{ at } \pm d$$

The eigenvalues may be characterised through the Rayleigh quotient:

$$\lambda_k = \inf_{(\phi, \cos\frac{i\pi y}{d}) = 0, i<k, \phi\neq 0} \frac{\int_{-d}^{d} \phi'^2\, dy}{\int_{d}^{d} \phi^2\, dy}$$

If we let $\phi = a \cdot \Psi_N$, we can characterise the minimum positive eigenvalue of $P^{-1}Q$:

$$\kappa_1^N = \min_{0 \neq a \notin \mathrm{span}\{e_0\}} \frac{a^T Q a}{a^T P a}$$

where $e_0 = (1, 0, \ldots, 0)$.

The following is well known, see [1]:

$$0 \le \kappa_k^N - \lambda_k \le C\{ \inf_{\substack{\chi = b\cdot\Psi_N \text{ for some b} \\ \phi \in M(\lambda_k) \\ \|\phi\| = 1}} \|\phi - \chi\| \}^2$$

where $M(\lambda_k)$ is the eigenspace corresponding to λ_k.

From these observations, we conclude two things:

- The localisation of the error estimator as defined in (4.5) can be founded on exponential decay of the solution away from "vertical" boundaries and/or rough spots in the load.
- A choice of basis functions is to be preferred over another if the first leads to a smaller minimum positive eigenvalue κ_1^N ($\kappa_0^N = 0 = \lambda_0$), since such a choice leads to the use of less basis functions (a smaller N_i) away from rough spots). That is evident from the following example.

There is an orthogonal matrix O such that $O^T P^{-1} Q O = D$, being diagonal. Setting $U = OV$ yields the following system of O.D.E.s

$$-V'' + \frac{1}{d^2}DV = O^T P^{-1} R = G$$

with the solution:

$$\begin{aligned} V_i(x) &= A_i \sinh(\sqrt{\kappa_i^N}\frac{x}{d}) + \frac{d}{2\sqrt{\kappa_i^N}} \times \{v_i^p(x)\} \\ v_i^p(x) &= e^{\sqrt{\kappa_i^N}\frac{x}{d}} \int_0^x e^{-\sqrt{\kappa_i^N}\frac{s}{d}} G_i\, ds + e^{-\sqrt{\kappa_i^N}\frac{x}{d}} \int_0^x e^{\sqrt{\kappa_i^N}\frac{s}{d}} G_i\, ds \end{aligned}$$

for $i \ge 1$. If one for example takes $\beta = \delta(x_0)$, the solution V_i involves terms of $\sinh(\sqrt{\kappa_i^N}\frac{x}{d})$ and $\cosh(\sqrt{\kappa_i^N}\frac{x-x_0}{d})/\sqrt{\kappa_i^N}$ where it becomes clear that a smaller κ_1^N improves localization.

For N = 2, choosing the basis *functions as in* (5.1) *yields* $\kappa_0^2 = 0, \kappa_1^2 = 15$, the latter approximating well the eigenvalue $\lambda_1 = \pi^2$ with respect to exponential decay. That is also the best one can do given the span for N = 2 (using $\{1, \eta^2\}$). In contrast, if one omits ϕ_0 as the first basis function, $\kappa_0^2 = 0 = \kappa_1^2$. For $N > 2$, the approximation of λ_1 can not get any worse. We therefore choose the basisfunctions as in (5.1).

Our initial computations suggest a practical confirmation and viability of many of the features described here of this method. It should be noted that we have not dealt with the issue whether or not this feed back method is adaptive, i.e. whether or not this feed back method is optimal with respect to some performance measure. It will be dealt with elsewhere.

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